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Distance-transitive Representations of Groups G with $PSL_2(q) \trianglelefteq G \leq P\Gamma L_2(q)$

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1. Introduction

Nowadays the most promising approach to the classification of distance-transitive graphs relies on the determination of the distance-transitive representations of finite groups. The main step within this approach is the determination of the primitive distance-transitive representations. In [19] this step was reduced to the following two problems:

(1) describe all distance-transitive representations of almost simple groups, i.e. the groups G with $N \trianglelefteq G \leq \mathrm{Aut}(N)$ for non-abelian simple groups N (*the almost simple case*);

(2) classify all distance-transitive graphs with automorphism groups having elementary abelian normal subgroups which act regularly on the vertex sets of the graphs (*the affine case*).

In view of the classification of the finite simple groups, in the almost simple case one should take for N one of the known non-abelian simple groups.

In [18] and [12] the almost simple case for $N \simeq A_n$ was treated completely, including the imprimitive representations. A natural next step in the almost simple case is the situation $N \simeq PSL_n(q)$ (distance-transitive representations of the linear groups).

The multiplicity-free primitive permutation representations of the linear groups for $n \geq 8$ are described in [9]. It was shown independently in [8] and [3] that among these representations only the representations on the cosets of maximal parabolic subgroups are distance-transitive. So, as the permutation character of a distance-transitive representation is multiplicity-free, each primitive distance-transitive graph on which a linear group of dimension $n \geq 8$ acts distance-transitively is a Grassmann graph. The primitive distance-transitive representations of the linear groups having small dimensions ($2 \leq n \leq 7$) are classified in [4]. Here, besides the Grassmann graphs, a number of sporadic examples exist.

General methods for determination of distance-transitive representations rely on the investigation of the multiplicity-free representations for the groups under consideration and of their subgroups having 'small' indexes. These methods are very effective when the groups are sufficiently 'large'. In the case of 'small' groups these methods usually do not enable one to obtain the final result. For example, for $N \simeq A_n$, $n \geq 18$ the description of the multiplicity-free permutational representations contained in [20] was used essentially both in [18] and [12]. But for small values of n some *ad hoc* arguments played a crucial role. For the linear groups one can see an analogous situation.

In [14] an algorithm for enumeration of the distance-transitive representations of a group using its character table is proposed. This algorithm is effective just for small groups. It was used by the authors for describing the distance-transitive representations of the non-abelian simple groups of order up to 10^8 and of some their extensions by outer automorphisms. As a consequence of that description it was proved that if N is one of the sporadic simple groups J_3, $O'N$ and He, and $N \trianglelefteq G \leq \mathrm{Aut}(N)$ then G has no

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distance-transitive representations. The description has also led to a conjecture that for sufficiently large q the only primitive distance-transitive representation of a group G with $PSL_2(q) \trianglelefteq G \leqslant P\Gamma L_2(q)$ is its natural doubly transitive representation of degree $q+1$.

For $G \simeq PSL_2(q)$ this conjecture was proved by L. F. Tchuda and the authors. Namely, [16] contains the description of all distance-transitive representations of the groups $PSL_2(q)$. In [15] the result was extended to the groups $PGL_2(q)$. The present paper contains the complete treatment of the case $PSL_2(q) \trianglelefteq G \leqslant P\Gamma L_2(q)$. The work was done independently of [4] but the latter paper had an influence on our exposition of the results. Note that we consider the imprimitive distance-transitive representations as well.

2. The Main Result and the General Scheme of the Proof

A few dozens of exceptional distance-transitive representations of the groups G with $PSL_2(q) \trianglelefteq G \leqslant P\Gamma L_2(q)$ for small values of q is known. These representations are collected in Table 1.

The notation in Table 1 is more or less standard. In particular $P\Sigma L_2(q)$ denotes the extension of the group $PSL_2(q)$ by the field automorphisms. The factor group $P\Gamma L_2(9)/PSL_2(9)$ is isomorphic to $Z_2 \times Z_2$, so there are three subgroups in $P\Gamma L_2(q)$

Table 1

q	G	H	n	$i(\Gamma)$	p.b.a	Γ
5	$PGL_2(5)$	S_4	5	$\{4; 1\}$	p.	K_5
	$PGL_2(5)$	D_{12}	10	$\{3, 2; 1, 1\}$	p.	Petersen graph O_3
	$PGL_2(5)$	A_4	10	$\{4, 3, 1; 1, 3, 4\}$	b., a.	$2.K_5$
	$PSL_2(5)$	Z_5	12	$\{5, 2, 1; 1, 2, 5\}$	a.	Icosahedron
	$PGL_2(5)$	D_8	15	$\{4, 2, 1; 1, 1, 4\}$	a.	Line graph of O_3
	$PGL_2(5)$	D_6	20	$\{3, 2, 2, 1, 1; 1, 1, 2, 2, 3\}$	b., a.	Desargues graph
7	$PSL_2(7)$	S_4	7	$\{6; 1\}$	p.	K_7
	$PSL_2(7)$	A_4	14	$\{12, 1; 1, 12\}$	a.	$\overline{7 \circ K_2}$
	$PGL_2(7)$	S_4	14	$\{3, 2, 2; 1, 1, 3\}$	b.	Heawood graph
	$PGL_2(7)$	D_{16}	21	$\{4, 2, 2; 1, 1, 2\}$	p.	Line graph of $PG(2, 2)$
	$PGL_2(7)$	D_{12}	28	$\{3, 2, 2, 1; 1, 1, 1, 2\}$	p.	Coxeter graph
8	$P\Gamma L_2(8)$	$Z_9 \lambda Z_6$	28	$\{27; 1\}$	p.	K_{28}
	$P\Gamma L_2(8)$	F_7^6	36	$\{14, 6; 1, 4\}$	p.	$J(9, 2)$
	$P\Gamma L_2(8)$	$Z_9 \lambda Z_3$	56	$\{27, 10, 1; 1, 10, 27\}$	a.	Gosset graph
9	$P\Gamma L_2(9)$	S_5	6	$\{5; 1\}$	p.	K_6
	$P\Gamma L_2(9)$	S_5	12	$\{6, 5; 1, 6\}$	b.	$K_{6,6}$
	$P\Gamma L_2(9)$	A_5	12	$\{5, 4, 1; 1, 4, 5\}$	b.a.	$2.K_6$
	$P\Gamma L_2(9)$	$S_4 \times Z_2$	15	$\{6, 4; 1, 3\}$	p.	$\overline{J(6, 2)}$
	M_{10}	$E_9 \lambda E_4$	20	$\{18, 1; 1, 18\}$	p.	$\overline{10 \circ K_2}$
	$P\Gamma L_2(9)$	$S_4 \times Z_2$	30	$\{3, 2, 2, 2; 1, 1, 1, 3\}$	b.	Tuttes 8-cage
	$P\Gamma L_2(9)$	$F_5^4 \times Z_2$	36	$\{5, 4, 2; 1, 1, 4\}$	b.	
	$P\Gamma L_2(9)$	SD_{16}	45	$\{4, 2, 2, 2; 1, 1, 1, 2\}$	p.	Gen. 8-gon
					p.	
11	$PSL_2(11)$	A_5	11	$\{10; 1\}$	p.	K_{11}
	$PGL_2(11)$	A_5	22	$\{5, 4, 3; 1, 2, 5\}$	b.	2-(11, 5, 2)-design
16	$P\Gamma L_2(16)$	$A_5 \cdot Z_2 \times Z_2$	68	$\{12, 10, 3; 1, 3, 8\}$	p.	Doro graph
17	$PSL_2(17)$	S_4	102	$\{3, 2, 2, 2, 1, 1, 1; 1, 1, 1, 1, 1, 1, 3\}$	p.	Biggs–Smith graph
19	$PSL_2(19)$	A_5	57	$\{6, 5, 2; 1, 1, 3\}$	p.	Perkel graph
25	$P\Sigma L_2$	$PGL_2(5) \times Z_2$	65	$\{10, 6. 4; 1, 2, 5\}$	p.	Locally Petersen

which contain $PSL_2(q)$ as a subgroup of index 2. These subgroups are $PGL_2(9)$, $P\Sigma L_2(9) \simeq S_6$ and M_{10}. For a graph Γ we denote by $\bar{\Gamma}$ the complementary graph of Γ, by $n \circ \Gamma$ the disjoint union of n copies of Γ, and by $2.\Gamma$ the bipartite double cover of Γ. Entries in column six indicate whether Γ is primitive (*p.*), bipartite (*b.*) and/or antipodal (*a.*). In some cases there are a few distance-transitive graphs corresponding to the same distance-transitive representation. In Table 1 we give only one graph for each distance-transitive representation. In addition, some graphs arise via representations of different groups under consideration. In such a case Table 1 contains the representation of the largest group.

The natural doubly transitive (degree $q+1$) representation of a group G with $PSL_2(q) \trianglelefteq G \leqslant P\Gamma L_2(q)$ is certainly a distance-transitive representation of G which corresponds to the complete graph K_{q+1}. Starting with this representation one can construct some imprimitive representations corresponding to antipodal covers of K_{q+1}. If G contains a subgroup of index 2 then G acts distance transitively on the bipartite double cover $2.K_{q+1}$. Another series of antipodal covers of K_{q+1} has the following description (cf. Proposition 12.5.3 in [5]).

Let $q = r.s+1$, where $r \geqslant 1$, $s \geqslant 1$ and let S be the subgroup of order s in the multiplicative group of $GF(q)$. Let W be a vector space of dimension 2 over $GF(q)$, provided with a non-degenerate symplectic form f. Let $\Delta(q, r)$ be the graph with vertex set $\{Sw \mid w \in W - \{0\}\}$, where $Sw \sim Sv$ if $f(w, v) \in S$. Then $\Delta(q, r)$ is an r-fold antipodal cover of K_{q+1}.

Now we can formulate the main result of the paper.

THEOREM 1. *Let G be a group with $PSL_2(q) \trianglelefteq G \leqslant P\Gamma L_2(q)$, where $q = p^m$, p is a prime, $q \geqslant 5$. Let G act distance-transitively on a graph Γ. Then one of the following holds*:

(i) *the distance-transitive representation of G on the vertex set of Γ is given in Table 1*;
(ii) *Γ is the complete graph K_{q+1}*;
(iii) *Γ is the bipartite double cover $2.K_{q+1}$ and G contains a subgroup of index 2*;
(iv) *Γ is the graph $\Delta(q, r)$, where $r \geqslant 2$, r divides $(q-1)/(2, (q-1))$ and either $r = 2$ or the following conditions (a)–(c) are satisfied*: (a) *r is a prime number*; (b) *m is divisible by $r-1$*; (c) *p is a primitive element modulo r.*

As mentioned in the introduction, the distance-transitive representations of the groups G with $PSL_2(q) \trianglelefteq G \leqslant P\Gamma L_2(q)$ for small q were studied by means of the algorithm presented in [14]: namely, it was shown that for $q \leqslant 100$ a distance-transitive representation of G is one of those described in Theorem 1. So we will deal here with the case $q > 100$.

In Section 3 we will show by standard arguments that all imprimitive distance-transitive representations of the groups under consideration can be constructed from the primitive ones by antipodal covering and bipartite doubling.

In Section 5 we will prove that, when $q > 100$, a primitive distance-transitive representation of a group G under consideration, is the natural doubly transitive representation of G. The proof relies on an application of some known properties of the subdegrees of distance-transitive representations reviewed in Section 4. A consequence of these properties is that the total number of subdegrees (i.e. the rank of the representation) is at most three times the number of different subdegrees. The subdegrees of the primitive permutation representations of the groups $PSL_2(q)$ were calculated in [21]. From these calculations one can conclude the following. Let t be the rank of a primitive representation of $PSL_2(q)$ and let m be the number of different subdegrees in this representation. Then m is bounded by a constant (for example,

$m<20$ holds). On the other hand, if q is large enough and the representation is not the doubly transitive one then $t \geqslant (\frac{1}{2})q^{\frac{1}{2}}$. So for $G = PSL_2(q)$ only for a finite number of q's can the above-mentioned condition on the subdegrees be satisfied. If we adjoin some outer automorphisms then m can increase and t can decrease, but not very much, since $|\mathrm{Out}(PSL_2(q))| \leqslant 2\log_2(q)$. Hence the general tendency is preserved. In order to obtain the strict bound $q \leqslant 100$ some particular situations should be considered in more detail (cf. Section 5).

To the end of the proof we will show in Section 6 that the imprimitive distance-transitive representations which arise from the natural doubly transitive representation are those listed in cases (ii) and (iv) of Theorem 1.

3. Reduction to the Primitive Case

First of all let us recall the main definitions. Let Γ be the graph that is finite and connected and let G be a subgroup of the automorphism group $\mathrm{Aut}(\Gamma)$ of Γ. The group G is said to act distance-transitively on Γ if the group G acts transitively on the pairs of vertices which are at distance i in Γ, for all $0 \leqslant i \leqslant d$. Here d is the diameter of the graph Γ. If G acts distance-transitively on Γ then Γ is a *distance-transitive graph* and the permutation group $(G, V(\Gamma))$ induced by G on the vertex set $V(\Gamma)$ of Γ is called the *distance-transitive representation* of G.

Let $PSL_2(q) \trianglelefteq G \leqslant P\Gamma L_2(q)$, $q \geqslant 5$ and G act distance-transitively on an imprimitive graph Γ. It is well known that an imprimitive distance transitive graph is either bipartite or antipodal or both (cf. cases (a)–(e) on p. 140 of [5]). If Γ is bipartite then the halved graph Γ' of Γ admits a distance-transitive action of a subgroup G' of G having index 2. So G' contains $PSL_2(q)$, and since the valency of Γ is at least 3, Γ' is not bipartite. Hence in this case we have a reduction.

Suppose that Γ is antipodal and not bipartite. Let $\bar{\Gamma}$ be the folded graph of Γ and let N be the kernel of G on $\bar{\Gamma}$. Suppose that $N \neq 1$. If the $d \geqslant 3$ then by Lemma 5.4 in [13] N should be elementary abelian, a contradiction. If Γ is antipodal of diameter 2, then Γ is complete multipartite and by our hypothesis the number of parts is at least 3. Using the description of subgroups in $PSL_2(q)$ due to L. Dickson (see Lemma 5.1) it is easy to show that G cannot act distance-transitively on such a graph.

This implies the following:

Proposition 3.1. *Let G be a group with $PSL_2(q) \trianglelefteq G \leqslant P\Gamma L_2(q)$. Then the imprimitive distance-transitive representations of G can be constructed from the primitive ones with the following two steps:*
(1) *antipodal covering of the graphs corresponding to the primitive representations;*
(2) *bipartite doubling of the primitive graphs and of the graphs constructed on the first step.*

4. Relations on the Subdegrees

In this section we present some known relations on the subdegrees of a distance-transitive representation and some corollaries of these relations.

Let Γ be a distance-transitive graph and $x \in V(\Gamma)$. For an integer i, $0 \leqslant i \leqslant d$, put $\Gamma_i(x) = \{y \mid d(x,y) = i\}$, where $d(x,y)$ is the distance between x and y in Γ. It is well known that the numbers

$$c_i = |\Gamma_1(y) \cap \Gamma_{i-1}(x)| \qquad \text{and} \qquad b_i = |\Gamma_1(y) \cap \Gamma_{i+1}(x)|$$

do not depend on the particular choice of x and y (see [1], [5] for details). Moreover,

$$1 = c_1 \leqslant c_2 \leqslant \cdots \leqslant c_d; \qquad b_0 \geqslant b_1 \cdots \geqslant b_{d-1} \tag{1}$$

and if $k_i = |\Gamma_i(x)|$ then

$$k_i = (b_0 \cdot b_1 \cdot \cdots \cdot b_{i-1})/(c_1 \cdot c_2 \cdot \cdots \cdot c_i). \tag{2}$$

The set of parameters $i(\Gamma) = \{b_0, b_1, \ldots, b_{d-1};\ c_1, c_2, \ldots, c_d\}$ is known as the *intersection array* of the graph Γ and it carries very significant information about the structure of Γ. If G acts distance-transitively on Γ then the numbers k_i, $0 \leqslant i \leqslant d$ are the subdegrees of the distance-transitive representation $(G, V(\Gamma))$, and $d+1$ is the rank of the representation. It follows from (1) and (2) that the following lemma holds:

LEMMA 4.1. *The sequence $k_0, k_1, \ldots, k_d$ is:*
(i) *unimodal, i.e. there exist integers s, t with $1 \leqslant s \leqslant t \leqslant d$ such that $k_0 = 1 < k_1 < \cdots < k_s = \cdots = k_t > \cdots > k_d$;*
(ii) *logarithmically convex, i.e. $k_i^2 \geqslant k_{i-1} \times k_{i+1}$ for all $1 \leqslant i \leqslant d-1$.*

The following lemma is a direct consequence of the main result of [11].

LEMMA 4.2. *If $k_1 > 2$ and $k_{s+1} \leqslant k_s$ then $d \leqslant 3 \cdot s$.*

Now, from Lemmas 4.1(i) and 4.2 we obtain the following:

COROLLARY 4.3. (i) *If $m = \#\{k_i \mid 0 \leqslant i \leqslant d\}$ then $d + 1 \leqslant 3 \cdot m$;*
(ii) *if for three distinct integers i, j, k we have $k_i = k_j = k_1$ then $k_i \geqslant k_s$ for all $0 \leqslant s \leqslant d$, i.e. k_i is the maximal subdegree of $(G, V(\Gamma))$.*

Let (G, V) be a distance-transitive representation of a group G and K be a normal subgroup of G such that (K, V) is a transitive permutation group. Let t be the rank of the group (K, V) and let m be the number of different subdegrees of this group. If $[G:K] = \alpha$ then since $K \trianglelefteq G$ each subdegree of the group (G, V) has the form $k \cdot \beta$, where k is a subdegree of (K, V) and β is a divisor of α. In particular, if t' is the rank of the representation (G, V) then $t' \geqslant t/\alpha$. Let $m(k)$ be the number of different lengths of suborbits of (G, V) which are unions of suborbits of (K, V) having length k. Then it is clear that $m(k) \leqslant \tau(\alpha)$, where $\tau(\alpha)$ is the number of divisors of α and $m(k) = 1$ if the multiplicity of k in (K, V) is 1 or 2. Since $t' = d + 1$, where d is the diameter of the corresponding distance-transitive graph, by Lemma 4.3(i) we have the following:

LEMMA 4.4. *Let G act distance-transitively on V. Let K be a normal subgroup of index α in G such that the permutation group (K, V) is transitive of rank t with m different subdegrees and s subdegrees have multiplicity 1 or 2. Then the following inequality holds:*

$$t/\alpha \leqslant 3(s + (m - s)\tau(\alpha)). \tag{3}$$

Let G, V, K be as above and let k_1, k_2 be different subdegrees of (K, V) with multiplicities e_1 and e_2 respectively. Suppose that $k_i/(k_1, k_2) > \alpha$ for $i = 1$ or 2. Then $k_1\beta_1 \neq k_2\beta_2$ for any divisors β_1, β_2 of α. Hence either for $i = 1$ or for $i = 2$ the subdegrees $k_i \cdot \beta$ are all non-maximal. By Lemma 4.1(i) the multiplicity of a non-maximal subdegree is at most 2. This implies the following:

LEMMA 4.5. *Let k_1 and k_2 be subdegrees of the group (K, V) with multiplicities e_1 and e_2 such that $k_i/(k_1, k_2) > \alpha$, $i = 1, 2$. Let $e = \min\{e_1, e_2\}$ and $\sigma(\alpha)$ be the sum of divisors of α. Then*

$$e \leqslant 2 \cdot \sigma(\alpha) \tag{4}$$

For each divisor β of α either $\beta \leqslant \alpha/2$ or $\beta = \alpha$. This gives the following trivial bounds:

$$\tau(\alpha) \leqslant (\alpha+2)/2, \qquad \sigma(\alpha) \leqslant (\alpha^2+6\alpha)/8. \tag{5.6}$$

5. The Case of Large q

The complete list of subgroups of the groups from the $PSL_2(q)$ series was determined by L. Dickson in [6]. Let $PSL_2(q) \trianglelefteq G \leqslant P\Gamma L_2(q)$, where q is a power of a prime number p. It is known that $P\Gamma L_2(q)$ is the full automorphism group of $PSL_2(q)$. If H is a maximal subgroup of G then $H \cap PSL_2(q)$ contains normalizers in $PSL_2(q)$ of all subgroups which are characteristic in $H \cap PSL_2(q)$. Keeping this fact in mind and using the list due to L. Dickson, it is easy to obtain a description of the maximal subgroups of the group G.

Lemma 5.1. *Let $PSL_2(q) \trianglelefteq G \leqslant P\Gamma L_2(q)$, q is a power of a prime number p and H be a maximal subgroup of G. Let $F = H \cap PSL_2(q)$. Suppose that $F \neq PSL_2(q)$. Then F is a subgroup from the following list*:
(i) *the normalizer of a Sylow p-subgroup of $PSL_2(q)$*;
(ii) *a subgroup A_4, S_4 or A_5, where $q = p$ in the former two cases—in the latter case either $q = p$, $p = \pm 1 \bmod 10$ or $q = p^2$, $p = \pm 3 \bmod 10$*;
(iii) *a dihedral subgroup of order $2(q-\varepsilon)/(2, q-1)$, where $\varepsilon \in \{1, -1\}$*;
(iv) *a subgroup isomorphic to $PSL_2(r)$, where $q = r^h$, h is an odd prime number, or a subgroup isomorphic to $PGL_2(r)$, where $q = r^2$.*
Moreover, the class of subgroups conjugated with F in G splits in $PSL_2(q)$ into at most two conjugacy classes.

The purpose of this section is to prove the following:

Proposition 5.2. *Let G, H and F be as above. Suppose that the representation of G on the cosets of H is distance-transitive. If $q > 100$ then F is the normalizer of a Sylow p-subgroup of $PSL_2(q)$.*

To prove Proposition 5.2 we should show that if F is one of the subgroups from (ii)–(iv) of Lemma 5.1 and $q > 100$ then the corresponding representation cannot be distance-transitive.

Firstly, let f be a subgroup from (ii). Here we consider only the case $F \simeq A_5$ (other cases are analogous to this one). If $q = \pm 1 \bmod 10$ then $PSL_2(q)$ contains exactly two conjugacy classes of maximal subgroups A_5 which are fused in $PGL_2(q)$. Hence $|H| = 60 \cdot n$ and $|G| \geqslant q(q^2-1)n/2$, where $n = 1$ or 2, if $n = 2$ then $q = p^2$, $p = \pm 3 \bmod 10$. On the other hand, it is known (cf. [12]) that if G acts distance-transitively on the cosets of H then $|G| < |H|^3$. So we have the following:

$$60^3 \cdot n^3 = |H|^3 > |G| \geqslant q(q^2-1)n/2.$$

If $n = 1$ then this inequality immediately implies that $q < 100$. If $n = 2$ then from the inequality we have $q < 169$. But between 100 and 169 there are no integers of the form p^2 where p is a prime such that $p = \pm 3 \bmod 10$. Thus we may assume that F is not a subgroup from (ii).

The consideration of the subgroups in (iii) and (iv) of Lemma 5.1 relies on a study of the subdegrees of corresponding representations. The subdegrees of all primitive representations of the groups from the $PSL_2(q)$ series were calculated in [21] on the base of a method proposed in [17] and [10]. These calculations can be extended to the primitive representations of groups G with $PSL_2(q) \trianglelefteq G \leqslant P\Gamma L_2(q)$. But if G contains field automorphisms then the result depends heavily on the numerical properties of q and the list of subdegrees becomes very complicated. So our strategy is as follows. First we study the subdegrees of the group $PSL_2(q)$ or $PGL_2(q)$. Then we apply Lemma 4.4 or Lemma 4.5 to reduce the problem to the consideration of a finite number of values of q. Finally, we calculate the subdegrees or a part of the subdegrees for these particular values of q.

We start with the representations over dihedral subgroups (case (iii) in Lemma 5.1). The rank of the representation of G on the cosets of H is at least $|G|/|H|^2$. In the situation under consideration if $q > 100$ then the rank is greater than 5. On the other hand, all dihedral subgroups of order $2(q-\varepsilon)/(2, q-1)$ for fixed $\varepsilon \in \{1, -1\}$ are conjugate in $PSL_2(q)$. So by a result from [2] we can assume that $G \simeq P\Gamma L_2(q)$. We will start with the subdegrees of $PGL_2(q)$ on the cosets of maximal dihedral subgroups and then apply Lemma 4.4. The subdegrees can be calculated either by the method used in [21], or by geometric arguments as was done in [4]. The result is presented in Table 2.

Now let us apply Lemma 4.4 to the case $G \simeq P\Gamma L_2(q)$, $N \simeq PGL_2(q)$, $q = p^n$. In this case $t \geqslant q/2$, $m - s = 1$, $s \leqslant 3$ and $\alpha = n$. So in view of (5) the inequality (3) has the following form:

$$p^2/2n \leqslant 3(3 + (n+2)/2), \qquad \text{i.e. } p^n \leqslant 3n^2 + 24n.$$

For prime powers $q > 100$ this inequality is satisfied only for $q = 2^8$ or 2^7.

If $q = 2^8$ then it is easy to see that there are at most five different subdegrees, and the rank is at least $2^8/2 \cdot 8 = 16$, a contradiction with Lemma 4.3(i).

For $q = 2^7$ the subdegrees are as follows:

$$S_{2(q-1)}\colon 1, ((2^7-1)7)^9, 2(2^7-1);$$
$$D_{2(q+1)}\colon 1, ((2^7+1)7)^9.$$

By Lemma 4.3(i) these subdegrees cannot come from a distance-transitive representation.

Now let us proceed to case (iv) in Lemma 5.1. Firstly let $H \cap PSL_2(q) = PSL_2(r)$, where $q = r^h$, h is an odd prime number and $r = p^n$. The subdegrees of the representation $PSL_2(r^h)$ on the cosets of $PSL_2(r)$ were calculated in [21]. The subdegrees are the following:

$$1, ((r^2-1)/\pi)^{\pi\alpha}, (r(r-1))^{\beta}, (r(r+1))^{\gamma}, (r(r^2-1)/\pi)^{\pi\delta},$$

where

$$\pi = (2, r-1),$$
$$\alpha = (r^{h-1}-1)/(r-1),$$
$$\beta = (r^h - r)/2(r+1),$$
$$\gamma = (r^h - r)/2(r-1),$$
$$\delta = \{r^{3h-2} - r^{h-2}(r^4 + r^3 + 2r^2 - r) + (r^3 + r^2 + r - 1)\}/(r^2-1)^2.$$

TABLE 2

	q	Rank	Subdegrees
$D_{2(q-1)}$	Even	$(q+2)/2$	$1, (q-1)^{(q-2)/2}, 2(q-1)$
$D_{2(q-1)}$	Odd	$(q+3)/2$	$1, (q-1)/2, (q-1)^{(q-3)/2}, 2(q-1)$
$D_{2(q+1)}$	Even	$q/2$	$1, (q+1)^{(q-2)/2}$
$D_{2(q+1)}$	Odd	$(q+1)/2$	$1, (q+1)/2, (q+1)^{(q-3)/2}$

It is easy to check that if $h \geq 5$ and $q > 100$ then $|G| > |H|^3$ and the representation cannot be distance-transitive [12]. So we can assume that $h = 3$. In this case the subdegrees are as follows:

$$1,\ ((r^2-1)/\pi)^{\pi(r+1)},\quad (r(r-1))^{r(r-1)/2},\quad (r(r+1))^{r(r+1)/2},\quad (r(r^2-1)/\pi)^{\pi(r^3+r-1)}.$$

The rank of the representation is $2r^3 + r^2 + 4r + 1$ if r is odd, and $r^3 + r^2 + 2r + 1$ if r is even. An application of Lemma 4.4 gives the inequality:

$$(2p^{3n} + p^{2n} + 4p^n + 1)/6n \leq 3(1 + 4(3n+1)) \qquad \text{if } q = p^{3n} \text{ is odd}$$

and

$$(p^{3n} + p^{2n} + 2p^n + 1)/3n \leq 3(1 + 4(3n+2)/2) \qquad \text{if } q = p^{3n} \text{ is even.}$$

For $q > 100$ these inequalities are satisfied only if $q = 5^3$ or $q = 2^9$.

A detailed analysis of subdegrees of $P\Gamma L_2(q)$ in these two particular cases shows that they do not correspond to distance-transitive representations.

Now let $H \cap PSL_2(q) = PGL_2(r)$, where $q = r^2$. Notice that in this case $G \leq P\Sigma L_2(q)$ since $PSL_2(q)$ contains two conjugacy classes of subgroups $PGL_2(r)$ which are fused in $PGL_2(q)$. In particular, $[G : PSL_2(q)] \leq 2n$ (recall that $q = (r^n)^2 = p^{2n}$). The subdegrees of $PSL_2(q)$ on the cosets of $PGL_2(r)$ are the following [21]:

$$1,\ (r^2-1),\ (r(r-1))^{(r-2)/2},\ (r(r+1))^{r/2} \qquad \text{if } r \text{ is even}$$

and

$$1,\ (r(r-\varepsilon)/2),\ (r^2-1),\ (r(r-1))^{(r-4-\varepsilon)/4},\ (r(r+1))^{(r-2+\varepsilon)/4} \tag{7}$$

if

$$r = 4s + \varepsilon, \quad \varepsilon = \pm 1.$$

Let us apply Lemma 2.5 for $k_1 = r(r-1)$, $k_2 = r(r+1)$. Notice that if $q > 100$ then $k_i/(k_1, k_2) \geq (r-1)/2 \geq 2m$. So in view of (6) the inequality (4) has the form

$$(2^n - 2)/2 \leq 2 \cdot \sigma(2n) \leq n^2 + 3n, \qquad \text{i.e. } 2^n \leq 2n^2 + 6n + 2 \quad \text{for } r = 2^n$$

and

$$(p^n - 5)/4 \leq 2 \cdot \sigma(2n) \leq n^2 + 3n, \qquad \text{i.e. } p^n \leq 4n^2 + 12n + 5 \quad \text{for odd } r.$$

For $q > 100$, $q = r^2$ these inequalities are satisfied iff

$$q \in \{2^{14}, 2^{12}, 2^{10}, 2^8, 3^8, 3^6, 5^4, p^2 \mid 11 \leq p \leq 19\}.$$

Let us calculate, for these particular values of q, the subdegrees for the group $P\Sigma L_2(q)$. For this purpose the following arguments are useful. Let $GF(q)^*$ be the multiplicative group of the field $GF(q)$, i.e. $GF(q)^*$ is cyclic of order $(q-1)$. Then $GF(q)^*$ can be considered as the complement in the normalizer of a Sylow p-subgroup in the group $PGL_2(q)$. For $\delta = \pm 1$ let Z_δ be the subgroup of order $(r+\delta)$ in $GF(q)^*$, $r^2 = q$. Then the suborbits of length $r(r-\delta)$ in (7) are indexed by unordered pairs $\{x, x^{-1}\}$, where x is an element of the factor group $GF(q)^*/Z_\delta$ and $x \neq x^{-1}$. So the fusion of these orbits in $P\Sigma L_2(q)$ is controlled by the action of the Frobenius automorphisms on the field $GF(q)$.

If α is a Frobenius automorphism of order 2 then it commutes with the subgroup of order $(r-1)$ in $GF(q)^*$ and inverts the elements of factor group $GF(q)^*/Z_{-1}$. Hence α acts trivially on the set of suborbits of length $r(r \pm 1)$. So if $q = p^2$ then the suborbits of $P\Sigma L_2(q)$ coincide with that of $PSL_2(q)$. Now it follows from Lemma 4.3(ii) and the subdegrees given in (7) that if $q = p^2$ then $p \leq 13$. For $p = 11$, 13 the subdegrees are as follows:

$$q = 11^2\text{: } 1,\ (11(11+1)/2),\ (11^2-1),\ (11(11-1))^2,\ (11(11+1))^2;$$

$$q = 13^2\text{: } 1,\ (13(13-1)/2),\ (13^2-1),\ (13(13-1))^2,\ (13(13+1))^2.$$

Since $a(a-1)a(a+1) > (a^2-1)^2$ for $a > 0$ the condition in Lemma 4.1(ii) can not be satisfied.

An application of Lemma 4.3(ii) to the cases of other q in the above list rules out all possibilities except $q = 3^6$. In the latter case the subdegrees are the following:

$$1, (3^3(3^2+1)/2), (3^6-1), (3^4(3^3-1))^2, (3^4(3^3+1))^2.$$

This case satisfies all of the conditions stated above and it needs more detailed study. It can be shown that this case does not lead to a distance-transitive graph. The easiest way to prove this fact is probably, to use the arguments from [4].

6. Antipodal Covers of Complete Graphs

Now let us describe the imprimitive distance-transitive representations of the groups under consideration, arising from their natural doubly transitive representations.

As above, let $PSL_2(q) \trianglelefteq G \leqslant P\Gamma L_2(q)$, $q = p^m$, $q \geqslant 5$ and let K denote the subgroup of G isomorphic to $PSL_2(q)$. In view of Lemma 3.1, the description will be carried out in two steps.

First, let Γ be antipodal and not bipartite. Suppose that Γ admits a distance-transitive action of G and that the folded graph $\bar{\Gamma}$ of Γ is the complete graph K_{q+1}, and that G induces on the vertex set of the latter graph the natural doubly transitive action. Let r be the size of an antipodal block of Γ. It was shown in [7] that a k- fold antipodal covering of K_{k+1} exists only if a Moore graph of valency k and diameter 2 exists; in particular, only if $k \in \{3, 7, 57\}$. In our situation this implies that $r < q$.

Let φ be the folding of Γ onto $\bar{\Gamma}$ which commutes with the action of G. Let $x \in V(\Gamma)$, $\bar{x} \in V(\bar{\Gamma})$ and $\varphi(x) = \bar{x}$. Let $G(x)$ and $G(\bar{x})$ be the stabilizers in G of the vertices x and $\bar{x}$, respectively. Then $E_q \cdot Z_{(q-1)/\varepsilon} \leqslant G(\bar{x}) \leqslant E_q \cdot Z_{q-1} \cdot Z_m$, where E_q denotes the elementary abelian group of order q (the additive group of the field $GF(q)$); Z_n is the cyclic group of order n and $\varepsilon = ((q-1), 2)$. On the other hand, $G(x)$ is a subgroup of index r in $G(\bar{x})$ and the permutation representation of $G(\bar{x})$ on the cosets of $G(x)$ is doubly transitive. This implies that $E_q \leqslant G(x)$. Now it is easy to see that the permutation group induced by the action of $G(\bar{x})$ on the cosets of $G(x)$ should be isomorphic to F_r^{r-1}; that is, the Frobenius group with cyclic kernel of order r and with the cyclic complement of order $r-1$. Moreover, r should divide $q-1$ and $r-1$ should divide m. Since the set of antipodal blocks is the only imprimitivity system of G on $V(\Gamma)$, K acts transitively on $V(\Gamma)$. This implies that $K(x) \simeq E_q \cdot Z_t$, where $t = (q-1)/(\varepsilon r)$, where ε is as above. This enables one to identify $V(\Gamma)$ with the vertex set of the graph $\Delta(q, r)$ defined before Theorem 1. Now, the distance-transitive of G on Γ leads to an isomorphism of Γ and $\Delta(q, r)$ and to the conditions (a)–(c) in Theorem 1. These conditions are equivalent to the claim that the automorphism group of $GF(q)$ acts transitively on the non-trivial elements of the factor group of the multiplicative group of $GF(q)$ over its subgroup S of order s.

Now let Γ be a bipartite graph such that G acts distance-transitive on Γ and the halved graph Γ' of Γ is either K_{q+1} or $\Delta(q, r)$ for some suitable q and r. Let $V(\Gamma) = U_1 \cup U_2$ be the bipartition of Γ and let G' be the subgroup of G which preserves the bipartition (G' has index 2 in G). Then it is easy to prove that the actions of G' on U_1 and U_2 are equivalent in the sense that the stabilizer in G' of a vertex in U_1 stabilizes a vertex in U_2. This implies that Γ' is primitive, and hence is isomorphic to K_{q+1} and that Γ is the bipartite double cover $2.K_{q+1}$.

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